Reflection principle and distribution of the running maximum of BM

Math 622

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1 The reflection principle

1.1 Definition

Let $W(t)$ be a Brownian motion w.r.t a filtration $\mathcal{F}(t)$ and τ a $\mathcal{F}(t)$ stopping time. We define

$$\begin{aligned} B^{\tau} &:= W(t), t \leq \tau \\ &:= W(\tau) - [W(t) - W(\tau)], t > \tau. \end{aligned}$$

That is B^{τ} is the same as $W(t)$ up to the random time τ and after time τ is obtained by reflecting $W(t)$ around the horizontal line $y = W(\tau)$. We say B^{τ} is a reflected BM at τ.

1.2 The reflection principle

Theorem 1.1. *The B^{τ} defined above is a $\mathcal{F}(t)$ Brownian motion.*

In words, the reflection principel says a refleted BM is a BM.
The heuristics of why the Theorem is true is
(i) The strong Markov property: $W(t) - W(\tau)$ is a Brownian motion independent of $\mathcal{F}(\tau)$
and
(ii) The negative of a BM is also a BM. Thus before t, B^{τ} is a BM, after τ it is also a BM (although starting at $W(\tau)$ instead of at 0). The key is how to show when we

go across τ the BM property is still preserved and we achieve that by Levy's characterization of BM.

Proof. Define

$$\begin{aligned} a(t) &= 1, t \leq \tau \\ &= -1, t > \tau. \end{aligned}$$

That is

$$\begin{aligned} a(t) &= \mathbf{1}_{t\leq\tau} - \mathbf{1}_{t>\tau} \\ &= \mathbf{1}_{t\leq\tau} - (1 - \mathbf{1}_{t\leq\tau}) \\ &= 2\mathbf{1}_{t\leq\tau} - 1. \end{aligned}$$

It is easy then to see $a(t) \in \mathcal{F}(t), \forall t$ since τ is a stopping time. It is also bounded, hence is in L^2. Thus we can consider $\int_0^t a(s)dW(s)$. We have

$$\begin{aligned} \int_0^t a(s)dW(s) &= \int_0^t 2\mathbf{1}_{s\leq\tau} dW(s) - W(t) \\ &= \int_0^t 2\mathbf{1}_{[0,\tau)}(s)dW(s) - W(t) \\ &= 2W(t\wedge\tau) - W(t) = B^\tau(t). \end{aligned}$$

(Just consider what happens when $\tau \leq t$ and $\tau > t$.)
Thus $B^\tau(t)$ is a martingale. Moreover, its quadratic variation is:

$$\langle B^\tau \rangle_t = \int_0^t \alpha^2(s)ds = t,$$

since $\alpha(s)$ is either 1 or -1. Thus by Levy's characterization, B^τ is a BM.

1.3 An important identity

Let $W(t)$ be a BM and $M(t) := \max_{[0,t]} W(s)$ its running maximum. The reflection principle helps us obtain the joint density between $W(t)$ and $M(t)$ through the following important identity:

$$\Big\{M(t) > m, W(t) < w\Big\} = \Big\{B(t) > 2m - w\Big\},$$

where $B(t) := B^{\tau_m}(t)$ is the BM obtained by reflecting $W(t)$ at time τ_m, the first hitting time of $W(t)$ to level m:

$$\tau_m := \inf\{t \geq 0 : W(t) = m\}.$$

See the picture accompanying this lecture note for illustration.

Remark 1.2. *Our goal with the identiy is to use it to derive the joint density $f_t(m, w)$ of $M(t), W(t)$, therefore we are only interested in considering $m \geq w$ and $m \geq 0$ because we always have $M(t) \geq W(t)$ and $M(t) \geq W(0) = 0$.*

Proof. Proof of the identity
(i) Suppose $M(t) > m$ and $W(t) < w$. Then $M(t) > m$ implies $\tau_m < t$ and hence

$$\begin{aligned} B(t) &= 2W(\tau_m) - W(t) \\ &= 2m - W(t) > 2m - w. \end{aligned}$$

(ii) Suppose $B(t) > 2m - w$. Then $B(t) > m$ because $w \leq m$. So it cannot be the case that $B(t) = W(t)$ since that would imply $W(t) > m$ and thus $\tau_m < t$, a contradiction to $B(t) = W(t)$ only when $t < \tau_m$. Thus $B(t) = 2m - W(t)$ and $\tau_m < t$ which implies $M(t) > m$. Moreover,

$$B(t) = 2m - W(t) > 2m - w$$

implies $W(t) < w$ and we are done.

1.4 Joint distribution of $W(t)$ and $M(t)$

From the identiy above and the reflection principle (which implies $B(t)$ is a BM) we have

$$P(M(t) > m, W(t) < w) = P(B(t) > 2m - w) = \int_{2m-w}^{\infty} \frac{e^{\frac{-x^2}{2t}}}{\sqrt{2\pi t}} dx.$$

If $f_t(m, w)$ is the joint density of $(M(t), W(t))$ then

$$P(M(t) > m, W(t) < w) = \int_{-\infty}^{w} \int_{m}^{\infty} f_t(z, x) dz dx = \int_{2m-w}^{\infty} \frac{e^{\frac{-x^2}{2t}}}{\sqrt{2\pi t}} dx.$$

Thus by the Fundamental Theorem of Calculus, we get

$$\begin{aligned} f_t(m, w) &= -\frac{\partial^2}{\partial m \partial w} P(M(t) > m, W(t) < w) \\ &= \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} \mathbf{1}_{m \geq 0, w \leq m}. \end{aligned}$$

2 Some explicit formula for $H_s(\alpha, \beta, k, b)$

Recall that when computing the price of Knockout Barrier and Lookback Options, we introduced the function

$$H_s(\alpha, \beta, k, b) := E\Big[\mathbf{1}_{\{W(s)\geq k\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)+\beta M(s)}\Big].$$

We will give the explicit formula for H_s in certain cases.

2.1 $H_s(\alpha, 0, k, b)$ when $0 \leq b \leq k$

Since $M(s) \geq W(s)$ we have if $W(s) \geq k$ then $M(s) \geq W(s) \geq k \geq b$.
Thus

$$\Big\{W(s) \geq k\Big\} \cap \Big\{M(s) \geq b\Big\} = \Big\{W(s) \geq k\Big\}.$$

In other words,

$$\mathbf{1}_{\{W(s)\geq k\}}\mathbf{1}_{\{M(s)>b\}} = \mathbf{1}_{\{W(s)\geq k\}}.$$

So

$$H_s(\alpha, 0, k, b) = E\Big[\mathbf{1}_{\{W(s)\geq k\}}e^{\alpha W(s)}\Big] = e^{s\frac{\alpha^2}{2}}N\big(\frac{s\alpha - k}{\sqrt{s}}\big). \tag{1}$$

2.2 $H_s(\alpha, 0, k, b)$ when $k < b$

Theorem 2.1. *If $k < b$,*

$$H_s(\alpha, 0, k, b) = e^{s\frac{\alpha^2}{2}}\Big\{N\big(\frac{s\alpha - b}{\sqrt{s}}\big) + e^{2\alpha b}\Big[N\big(\frac{-s\alpha - b}{\sqrt{s}}\big) - N\big(\frac{-s\alpha - 2b + k}{\sqrt{s}}\big)\Big]\Big\}.$$

Proof. Since $k < b$,

$$\begin{aligned} E\Big[\mathbf{1}_{\{W(s)\geq k\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)}\Big] &= E\Big[\mathbf{1}_{\{W(s)\geq b\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)}\Big] \\ &\quad + E\Big[\mathbf{1}_{\{k\leq W(s)<b\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)}\Big]. \end{aligned}$$

Now

$$E\Big[\mathbf{1}_{\{W(s)\geq b\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)}\Big] = H_s(\alpha, 0, b, b),$$

and we have found the expression for $H_s(\alpha, 0, b, b)$ in Section 2.1. As for the 2nd term, observe that

$$\Big\{k < W(s) < b, M(s) > b\Big\} = \Big\{W(s) < b, M(s) > b\Big\} \cap \Big\{k < W(s), M(s) > b\Big\}.$$

We have showed that

$$\Big\{W(s) < b, M(s) > b\Big\} = \Big\{B^{\tau_b}(s) > b\Big\},$$

where B^{τ_b} is again $W(t)$ reflected at τ_b, the first hitting time of $W(t)$ to level b. We claim that

$$\Big\{k < W(s), M(s) > b\Big\} = \Big\{M(s) > b, B^{\tau_b}(s) < 2b - k\Big\}.$$

(This is left as part of the homework).
Thus noting that $B^{\tau_b}(s) > b$ implies $M(s) > b$ we get

$$\begin{aligned}\Big\{k < W(s) < b, M(s) > b\Big\} &= \Big\{B^{\tau_b}(s) > b\Big\} \cap \Big\{M(s) > b, B^{\tau_b}(s) < 2b - k\Big\} \\ &= \Big\{b < B^{\tau_b}(s) < 2b - k\Big\}.\end{aligned}$$

We leave it as the other part of the homework to use this and (1) to complete the proof.

2.3 $H_s(\alpha, \beta, -\infty, b)$

Theorem 2.2.

$$\begin{aligned}H_s(\alpha, \beta, -\infty, b) &= \frac{\beta + \alpha}{\beta + 2\alpha} 2e^{\frac{(\alpha+\beta)^2}{2}s} N\big(\frac{(\alpha + \beta)s - b}{\sqrt{s}}\big) \\ &\quad + \frac{2\alpha}{\beta + 2\alpha} e^{\frac{\alpha^2}{2}s} e^{b(\beta+2\alpha)} N\big(-\frac{\alpha s + b}{\sqrt{s}}\big).\end{aligned}$$

Proof. See Ocone's Lecture 5 part 2 proof, page 3.